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# Gap Number of Groups (Generic structures and their applications)

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## Gap Number of Groups

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Finite gap number is introduced by J. C. Lennox and J. E. Roseblade in [LR].  
We study groups of the small gap number.

### 1 ladder index

Let  $T$  be complete theory in an  $L$ ,  $\phi(\bar{x}, \bar{y})$   $L$ -formula ( $\bar{x}, \bar{y}$  are free variables).

**Definition 1** An  $n$ -ladder for  $\phi$  is a sequence  $(\bar{a}_0, \dots, \bar{a}_{n-1}; \bar{b}_0, \dots, \bar{b}_{n-1})$  of tuples in some model  $M$  of  $T$ , such that

$$\forall i, j < n, M \models \phi(\bar{a}_i, \bar{b}_j) \Leftrightarrow i \leq j.$$

We say that  $\phi$  is **stable** formula if there exists  $n$  such that no  $n$ -ladder for  $\phi$  exists; otherwise it is **unstable**. The least such  $n$  is the **ladder index** of  $\phi$ .

**Theorem 2** The theory  $T$  is unstable if and only if there exists an unstable formula in  $L$  for  $T$ .

Henceforth we consider the ladder index for the commutativity formula “ $xy = yx$ ”. The ladder index of a group  $G$  for the commutativity formula is denoted by  $\ell(G)$ .

## 2 gap number

Let  $G$  be a group.

**Definition 3** A group  $G$  has a finite gap number if for any subgroups  $H_0, H_1, \dots, H_n, \dots$  of  $G$ , among the sequence

$$C_G(H_0) \leq C_G(H_1) \leq \dots \leq C_G(H_n) \leq \dots$$

there exist at most  $m$  many strict inclusions. The most such  $m$  is the **gap number** of  $G$ , and denoted by  $g(G)$ .

**Lemma 4** Let  $g(G) = n$ . Suppose that the sequence

$$C_G(H_0) > C_G(H_1) > \dots > C_G(H_n)$$

gives gap number  $n$ . Then there exists  $a_i (0 \leq i \leq n)$  in  $G$  such that  $C_G(H_i) = C_G(\{a_0, a_1, \dots, a_i\})$  for each  $i$ . In particular we may do  $a_0 = 1$ . Henceforth we abbreviate as  $C_G(\{a_0, a_1, \dots, a_i\}) = (a_0, a_1, \dots, a_i)$ .

By Lemma 4, we can prove the following:

**Theorem 5** [ITT]  $\ell(G) = g(G) + 2$ .

**Lemma 6** Let  $A, B \subset G$  with  $A \subset B$ . Then  $(A) \supset (B)$ .

**Lemma 7** Let  $A \subset G$ . Then  $((A)) = (A)$ .

By the above lemma, the following holds:

**Lemma 8** Let  $g(G) = n$ . Suppose  $(a_0, \dots, a_n; b_0, \dots, b_n)$  is  $(n+1)$ -ladder. Then

$$\begin{aligned} ((a_0)) &= (b_n, \dots, b_1, b_0); \\ ((a_0, a_1)) &= (b_n, \dots, b_1); \\ &\vdots \\ ((a_0, \dots, a_n)) &= (b_n). \end{aligned}$$

**Lemma 9** Let  $g(G) = n$ . Suppose that the sequence

$$G > (a_1) > \dots > (a_1, a_2, \dots, a_n)$$

gives gap number  $n$ . Then  $(a_1, a_2, \dots, a_{n-1})$  is abelian.

### 3 Groups of gap number up to four

From now on we do not consider the ladder index but the gap number.

**Theorem 10** [ITT]  $g(G) = 0$  if and only if  $G$  is abelian.

**Theorem 11** [ITT] There exist no groups  $G$  of  $g(G) = 1$ .

**(proof)** Let  $g(G) \geq 1$ . Then there exists  $a \in G$  such that  $G > (a)$ . Since  $(a) \neq G$ , there exists  $b \notin (a)$ . Therefore, we have  $G > (a) > (a, b)$ . Thus  $g(G) \geq 2$ .

**Theorem 12** [ITT]  $g(G) = 2$  if and only if  $G$  is not abelian, and for any  $a, b \in G \setminus Z(G)$ , if  $(a) \neq (b)$  then  $(a, b) = Z(G)$ .

**Example 13**  $g(S_3) = g(D_n) = 2$  ( $D_n$  is a dihedral group).

**Example 14**  $g(SL(2, F)) = 2$  ( $F$  is a field).

**Theorem 15** [ITT] There exist no groups  $G$  of  $g(G) = 3$ .

**(proof)** Let  $g(G) \geq 3$ . Then there exist  $a_1, a_2 \in G$  such that  $G > (a_1) > (a_1, a_2) > Z(G)$ .

**Case 1:**  $a_1 a_2 = a_2 a_1$ .

Since  $(a_1) \neq (a_2)$ , we may assume  $(a_1) \setminus (a_2) \neq \emptyset$ . Let  $b \in (a_1) \setminus (a_2)$ . As  $a_1 \notin G$ , there exists a  $c \in G \setminus (a_1)$ . Therefore, we have

$$G > (a_1) > (a_1, a_2) > (a_1, a_2, b) > (a_1, a_2, b, c).$$

Thus  $g(G) \geq 4$ .

**Case 2:**  $a_1 a_2 \neq a_2 a_1$ .

There exists a  $d \in (a_1, a_2) \setminus Z(G)$ . Since  $d \notin Z(G)$ , we can find  $e \notin G \setminus (d)$ . Then we have

$$G > (d) > (d, a_1) > (d, a_1, a_2) > (d, a_1, a_2, e).$$

Thus  $g(G) \geq 4$ .

**Example 16**  $g(S_4) = g(S_5) = 4$ .

## 4 Groups of gap number five

In this section, we investigate whether a group  $G$  of gap number 5 exists or not.

Let  $g(G) = 5$  and let  $(1, a_1, \dots, a_5; b_0, \dots, b_4, 1)$  be 6-ladder.

**Case 1:**  $a_1a_2 = a_2a_1, a_1a_3 = a_3a_1$  and  $a_2a_3 = a_3a_2$ .

Then we have

$$G > (a_1) > (a_1, a_2) > (a_1, a_2, a_3) > (a_1, a_2, a_3, b_2) > (a_1, a_2, a_3, b_2, b_1) > Z(G).$$

Thus,  $g(G) \geq 6$ .

**Case 2:**  $a_1a_2 = a_2a_1, a_1a_3 = a_3a_1, a_2a_3 \neq a_3a_2$  and  $a_1a_4 \neq a_4a_1$ .

Then we have

$$G > (b_4) > (b_4, a_1) > (b_4, a_1, a_2) > (b_4, a_1, a_2, a_3) > (b_4, a_1, a_2, a_3, a_4) > Z(G).$$

Thus,  $g(G) \geq 6$ .

**Case 3:**  $a_1a_2 = a_2a_1, a_1a_3 = a_3a_1, a_2a_3 \neq a_3a_2$  and  $a_1a_4 = a_4a_1$ .

Then we have

$$G > (a_1) > (a_1, b_3) > (a_1, b_3, a_2) > (a_1, b_3, a_2, a_3) > (a_1, b_3, a_2, a_3, a_4) > Z(G).$$

Thus,  $g(G) \geq 6$ .

**Case 4:**  $a_1a_2 = a_2a_1, a_1a_3 \neq a_3a_1$  and  $a_2a_3 = a_3a_2$ .

Then we have

$$G > (a_2) > (a_2, a_1) > (a_2, a_1, a_3) > (a_2, a_1, a_3, a_4) > Z(G).$$

Moreover  $a_2a_1 = a_1a_2, a_2a_3 = a_3a_2$  and  $a_1a_3 \neq a_3a_1$ . By case 2, 3,  $g(G) \geq 6$ .

**Case 5:**  $a_1a_2 = a_2a_1$  and  $a_1a_3 \neq a_3a_1$ .

Then we have

$$G > (b_4) > (b_4, b_3) > (b_4, b_3, a_1) > (b_4, b_3, a_1, b_1) > Z(G).$$

Moreover  $b_4a_1 = a_1b_4$  and  $b_3a_1 = a_1b_3$ . By case 1, 4,  $g(G) \geq 6$ .

Therefore, by case 1 through 5, we hold  $a_1a_2 \neq a_2a_1$ .

**Case 6:**  $a_1a_2 \neq a_2a_1$  and  $a_1a_3 = a_3a_1$ .

Then we have

$$G > (a_1) > (a_1, a_3) > (a_1, a_3, a_2) > (a_1, a_3, a_2, a_4) > Z(G).$$

Moreover  $a_1a_3 = a_3a_1$ . Thus,  $g(G) \geq 6$ .

**Case 7:**  $a_1a_2 \neq a_2a_1$ ,  $a_1a_3 \neq a_3a_1$  and  $a_2a_3 = a_3a_2$ .

Then we have

$$G > (a_2) > (a_2, a_1) > (a_2, a_1, a_3) > (a_2, a_1, a_3, a_4) > Z(G).$$

Moreover  $a_2a_3 = a_3a_2$ . By case 6,  $g(G) \geq 6$ .

Therefore, by case 1 through 7, we hold  $a_1a_2 \neq a_2a_1$ ,  $a_1a_3 \neq a_3a_1$  and  $a_2a_3 \neq a_3a_2$ .

**Case 8:** all of  $a_1, a_2, a_3, a_4$  are noncommutative except  $a_1a_4 = a_4a_1$ ,  $a_2a_4 = a_4a_2$ .

Then we have

$$G > (a_1) > (a_1, a_2) > (a_1, a_2, a_4) > (a_1, a_2, a_4, a_3) > Z(G).$$

Moreover  $a_1a_4 = a_4a_1$ . By case 6,  $g(G) \geq 6$ .

**Case 9:** all of  $a_1, a_2, a_3, a_4$  are noncommutative except  $a_1a_4 = a_4a_1$ ,  $a_3a_4 = a_4a_3$ .

Then we have

$$G > (a_1) > (a_1, a_3) > (a_1, a_3, a_4) > (a_1, a_3, a_4, a_2) > Z(G).$$

Moreover  $a_1a_4 = a_4a_1$ . By case 6,  $g(G) \geq 6$ .

**Case 10:** all of  $a_1, a_2, a_3, a_4$  are noncommutative except  $a_2a_4 = a_4a_2$ ,  $a_3a_4 = a_4a_3$ .

Then we have

$$G > (a_2) > (a_2, a_3) > (a_2, a_3, a_4) > (a_2, a_3, a_4, a_1) > Z(G).$$

Moreover  $a_2a_4 = a_4a_2$ . By case 6,  $g(G) \geq 6$ .

In the cases of remaining we understand the following:

**Lemma 17** *Let  $G > (a_1) > (a_1, a_2) > \cdots > (a_1, a_2, a_3, a_4, a_5) = Z(G)$ . Then we can do as follows: all of  $a_1, a_2, a_3, a_4$  are noncommutative except  $a_1a_4 = a_4a_1, a_2a_4 = a_4a_2, a_3a_4 = a_4a_3, a_1a_5 = a_5a_1, a_2a_5 = a_5a_2, a_3a_5 = a_5a_3$ .*

**(proof)** We have

$$G > (a_1) > (a_1, a_2) > (a_1, a_2, a_3) > (a_1, a_2, a_3, b_4) > (a_1, a_2, a_3, b_4, b_3).$$

Moreover all of  $a_1, a_2, a_3$  are noncommutative, and all of  $b_4, b_3, b_2$  are noncommutative, as desired.

**Question 18** *Does there exist a group  $G$  of  $g(G) = 5$ ?*

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